Central Conicoids

The conicoids having centre is called Central Conicoids

General equation —

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$

standard equation

$ax^2+by^2+cz^2=1$ (whose centre is 0,0,0)

Equation of Ellipsoid — It is a closed surface.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Hyperboloid of one sheet — Its section is defined by a plane ∥ to $x=0$ or $y=0$ is hyperbola.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Hyperboloid of two sheet — Its section is defined by the plane ∥ to $y=0$ or $z=0$ is hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Tangent line — A line which meets the conicoid in two coincident points is called the tangent line to the conicoid.

To find the equation of tangent plane of the conicoid.

Let $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ be a st line which cuts the conicoid $ax^2+by^2+cz^2=1$ then we get a quadratic equation in r. which cuts it at two points. ie

$r^2(al^2 + bm^2 + cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) +$

$a\alpha^2 + \beta b^2 + c\gamma^2 - 1 \quad = 0.$

If (α, β, γ) lies on the conicoid then $a\alpha^2 + b\beta^2 + c\gamma^2 = 1$

If $al\alpha + bm\beta + cn\gamma = 0$ then the give line is tangent to the conicoid at (α, β, γ).

Now eliminating l, m, n as put $l = (x - \alpha) \cdots$

ie $a(x-\alpha)\alpha + b(y-\beta)\beta + c(z-\gamma)\gamma = 0$

$\Rightarrow \quad ax\alpha + by\beta + cz\gamma = 1$

is the required equation of the tangent plane.

To find the condition of tangency.

A plane $lx + my + nz = p$ must touch the conicoid

$ax^2 + by^2 + cz^2 = 1$ if

$$\frac{l^2}{a} = \frac{m^2}{b} = \frac{n^2}{c} = p^2$$

is the condition of tangency at (α, β, γ).

Equation of tangent plane is

$$lx + my + nz = \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

Point of contact of tangent plane is

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

Example Find the equation of a tangent planes to the conicoid $7x^2 - 3y^2 - z^2 + 21 = 0$ which passes through the line. $7x - 6y + 9 = 0$; $z = 3$.

Sol. Any plane through the given line is

$$7x - 6y + 9 + \lambda(z - 3) = 0$$

i.e $$7x - 6y + \lambda z + 3(3 - \lambda) = 0 \quad \text{— (1)}$$

Equation of the conicoid is

$$-7x^2 + 3y^2 + z^2 = 21$$

or $$\frac{x^2}{-3} + \frac{y^2}{7} + \frac{z^2}{21} = 1 \quad \text{— (2)}$$

If (1) touches (2) then by $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

$$-3(7)^2 + 7 \times (-6)^2 + \lambda^2 \times 21 = [3(3-\lambda)]^2$$

$$-147 + 252 + 21\lambda^2 = 81 - 54\lambda + 9\lambda^2$$

$$105 + 21\lambda^2 = 81 - 54\lambda + 9\lambda^2$$

$$35 + 7\lambda^2 = 27 - 18\lambda + 3\lambda^2$$

$$4\lambda^2 + 18\lambda + 8 = 0$$

$$2\lambda^2 + 9\lambda + 4 = 0$$

$$\lambda = -4, -\frac{1}{2}$$

Put the values of λ in (1) we get

$\therefore$ equation of tangent planes are

$$7x - 6y - 4z + 21 = 0$$

$$14x - 12y - z + 21 = 0.$$

Director sphere of a Conicoid.

The locus of point of intersection of three mutually ⊥ tangent planes to a conicoid is called the director sphere of a conicoid

Let the three mutually ⊥ tangent planes to the given conicoid be

$$l_1 x + m_1 y + n_1 z = \sqrt{\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}} \quad \text{—(i)}$$

$$l_2 x + m_2 y + n_2 z = \sqrt{\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}} \quad \text{—(ii)}$$

$$l_3 x + m_3 y + n_3 z = \sqrt{\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}} \quad \text{—(iii)}$$

Here $\Sigma l_1^2 = \Sigma l_2^2 = \Sigma l_3^2 = 1$

and $l_1 m_1 + l_2 m_2 + l_3 m_3 = m_1 n_1 + \dots = n_1 l_1 + \dots = 0$

squaring and adding (i) (ii) and (iii)

we get $\quad x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

ex Find the equation of the tangent plane to the conicoid $2x^2 - 6y^2 + 3z^2 = 5$ at $(1, 0, -1)$.

Sol. equation of tangent plane of $2x^2 - 6y^2 + 3z^2 = 5$

is $\quad 2\alpha x - 6\beta y + 3\gamma z = 5$

ie $\quad 2 \times 1 \times x - 6 \times 0 \times y + 3 \times -1 \times z = 5$

$\quad 2x - 3z = 5$

Hence the required line is

$$y+1 = 0 = -x+2y+z$$

Diameter and Diametral planes —

Any chord of the central conicoid passing through the centre is called the diameter.

A plane which bisects a system of parallel chords is called a diametral plane of the conicoid.

Section with given Centre.

Let (α, β, γ) be the middle point of any chord

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

and $ax^2+by^2+cz^2=1$ be the conicoid.

Then equation of the chords which bisected by (α, β, γ) is $T = S_1$

ie $a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$

Equation of a diametral plane

Let (α, β, γ) be the middle point (say centre). Then

$$alx + bmy + cnz = 0$$

is the equation of diametral plane.

or It is locus of middle points of the system of parallel chords of the conicoid.

Normal – A line passing through a point P on a surface perpendicular to the tangent plane at P is called the normal to the surface at P.

equation of normal to ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x - x'}{x'/a^2} = \frac{y - y'}{y'/b^2} = \frac{z - z'}{z'/c^2}$$

Result → There are six points on ellipsoid the normals at which passes through a given point (α, β, γ).

equation of plane of contact, Polar plane: $a\alpha x + by\beta + c\gamma z = 1$

Pole of a plane is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

Equation of polar line w.r.t. the conicoid $ax^2 + by^2 + cz^2 = 1$

$$a\alpha x + b\beta y + c\gamma z - 1 = 0 \; ; \; alx + bmy + cnz = 0.$$

ex: Find the equation of the polar of the line $-2x = 25y - 1 = 2z$ with respect to the conicoid $2x^2 - 25y^2 + 2z^2 = 1$

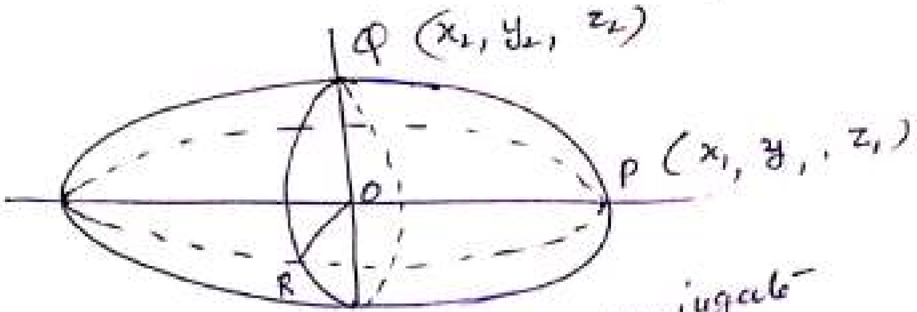
eq of the given line $\frac{x}{-1/2} = \frac{25y - 1}{1} = \frac{z}{1/2}$

now $a\alpha x + b\beta y + c\gamma z - 1 = 0 + 25 \times \left(\frac{-1}{25}\right) y + 0 - 1 = 0$

ie $y + 1 = 0$

$alx + bmy + cnz = -x + 2y + z = 0$

Conjugate diameters and Diametral planes.



QOR, ROP and POQ are the conjugate diametral planes

OP, OQ, OR are the conjugate diameters.

let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be ellipsoid.

The diametral plane of the line OP is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 0$

at Q it will be $\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{z_1 z_2}{c^2} = 0$

ie the point P lies on the plane $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} = 0$

which is the diametral plane of OQ.

ie Q is on diametral plane of OP and P is on the diametral plane of OQ.

Similar relation holds for other point P, R & Q, R

1. The three semi diameters of an ellipsoid, which are such that the plane containing any two of them is the diametral plane of the third, are called the conjugate semi diameters of the ellipsoid.
2. The three diametral planes, which are such that each is the diametral plane of the line of intersection of other two, are called the conjugate diametral planes of ellipsoid

Propositions

(i) Sum of the squares of the lengths of three conjugate semi diameters of an ellipsoid is constant

ie $OP^2 + OQ^2 + OR^2 = \text{constant}$

(ii) The volume of parallelopiped formed by three conjugate diameters (semi) as coterminous edge is constant.

ie 6 x volume of tetrahedron OPQR = constant

(iii) Sum of squares of the areas of the faces of the parallelopiped is constant.

$$A_1^2 + A_2^2 + A_3^2 = \text{constant}.$$

ex Find the equation of the plane which cuts the conicoid $x^2 + 4y^2 - 5z^2 = 1$ in a conic whose centre is the point $(2,3,4)$.

Sol The required equation is $T = S_1$

$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$$

$$2x + 12y - 20z = 4 + 36 - 5\times 16$$

ie $x + 6y - 10z + 20 = 0$.

ex Find the centre of the conic given by the equation

$$3x^2 + 2y^2 - 5z^2 = 4, \quad 2x - 2y - 5z + 5 = 0$$

Sol. Equation using $T = S_1$ we get at (α, β, γ)

$$3x\alpha + 2y\beta - 5z\gamma = 3\alpha^2 + 2\beta^2 - 5\gamma^2$$

Comp. $\frac{3\alpha}{2} = \frac{\beta}{-1} = \frac{\gamma}{-1} = \frac{3\alpha^2 + 2\beta^2 - 5\gamma^2}{-5}$

on solving we get $(-2, 3, -1)$.